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Product integration methods for solving a system of nonlinear Volterra integral equations

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Abstract

In this paper the technique of subtracting out singularities is used to derive explicit and implicit product Euler schemes with order one convergence and a product trapezoidal scheme with order two convergence for a system of Volterra integral equations with a weakly singular kernel. The convergence proofs of the numerical schemes are presented; these are nonstandard since the nonlinear function involved in the integral equation system does not satisfy a global Lipschitz condition.

Keywords: Volterra equations; Product integration methods

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1. Introduction

Consider the following heat equation with nonlinear and nonlocal boundary conditions:

$$u_t(x,t) = u_{xx}(x,t), \quad 0 < x < 1, \ t > 0, \tag{1.1a}$$

$$u(x,0) = 1, \quad 0 < x < 1, \tag{1.1b}$$

$$u_x(0,t) = 0, \quad t > 0, \tag{1.1c}$$

$$u_x(1,t) = \frac{Em}{1+L}\{L\gamma(t) - (1-\gamma(t))u(1,t)\}, \quad t > 0, \tag{1.1d}$$

$$m\gamma(t) + \int_0^1 u(x,t)\,\mathrm{d}x = 1, \quad t > 0, \tag{1.1e}$$

where E, L, m are positive constants.

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This equation arises from the modelling of a particular type of diffusion–reaction system (see [1, 4, 8, 10, 11]).

By using Laplace transforms (see [4, 10]) the initial-boundary value problem (1.1) can be transformed into the following nonlinear integro-differential equation:

$$\gamma'(t) = -C\left(L\gamma(t) - (1-\gamma(t))\left(1 - m\int_0^t k(t-s)\gamma'(s)\,\mathrm{d}s\right)\right), \quad t > 0, \tag{1.2a}$$

with

$$\gamma(0) = 0, \tag{1.2b}$$

where

$$k(t) = \frac{1}{\sqrt{\pi t}}\left(1 + 2\sum_{n=1}^{\infty} \exp(-n^2/t)\right), \quad t > 0, \tag{1.3}$$

and

$$C = E/(1+L). \tag{1.4}$$

The value of $u(1,t)$ can be obtained (see [4, 10]) through

$$u(1,t) = 1 - m\int_0^t k(t-s)\gamma'(s)\,\mathrm{d}s, \quad t > 0. \tag{1.5}$$

The integro-differential equation (1.2) can be solved numerically using an explicit Euler product integration scheme (the method and its convergence proof can be found in [8]). This permits an efficient calculation of $u(1,t)$ and through (1.1d) the evaluation of $u_x(1,t)$, which is often the quantity required by electrochemists as it represents the flux to the surface at $x = 1$. However, the asymptotic expansion of $\gamma(t)$ at $t = 0$ (see [8]) displays a $t^{3/2}$ singularity and so a direct application of any product integration method or collocation method to the integro-differential equation (1.2) will result in a global convergence rate of $\frac{1}{2}$ (see e.g. [3, 18]).

For the initial-boundary value problem (1.1) Jumarhon and McKee [11] set up the following equivalent system of Volterra integral equations:

$$\phi_1(t) = 1 + \int_0^t k(t-s)F(\phi_1(s), \phi_2(s))\,\mathrm{d}s, \tag{1.6a}$$

$$\phi_2(t) = 1 + \int_0^t F(\phi_1(s), \phi_2(s))\,\mathrm{d}s, \tag{1.6b}$$

where

$$\phi_1(t) = u(1,t), \qquad \phi_2(t) = \int_0^1 u(x,t)\,\mathrm{d}x, \tag{1.7}$$

and

$$F(\phi_1(t), \phi_2(t)) = C(L - (m-1)\phi_1(t) - L\phi_2(t) - \phi_1(t)\phi_2(t)). \tag{1.8}$$

In the same paper it was proved that system (1.6) has a unique continuous solution $(\phi_1(t), \phi_2(t))$ on $[0, \infty)$, and the asymptotic values of this solution were obtained for $t \to \infty$.

In this paper we derive explicit and implicit product Euler schemes and a product trapezoidal scheme for system (1.6) (order three and higher order product integration schemes can be derived similarly); we demonstrate that the Euler schemes converge with order one while the trapezoidal scheme converges with order two. We also derive asymptotic expansions for $\phi_1(t)$ and $\phi_2(t)$ as $t \to 0$ and note that these display singularities at $t = 0$. In the derivation of the methods we use these asymptotic expansions to subtract out the singularity. It is found that the convergence proofs for the product integration schemes are not straightforward because the nonlinear function F does not satisfy a global Lipschitz condition.

2. Asymptotic expansion and numerical schemes

2.1. Asymptotic expansions

It is important to look at the asymptotic expansions of the solutions of Abel–Volterra integral equations because the degree of the singularity of the integrand affects the order of the convergence of any product integration method. Miller and Feldstein [14] and more recently Lubich [13] studied the nonsmoothness behaviour of solutions of systems of Abel–Volterra integral equations of the second kind. In our case, we can apply the same argument as in [13] to show that the integral equation system (1.6) has an asymptotic solution of the form

$$\phi_1(t) = \sum_{i=0}^{\infty} a_i t^{i/2}, \qquad \phi_2(t) = \sum_{i=0}^{\infty} b_i t^{i/2},$$

near $t = 0$. Thus, for small t noting that $\sum_{n=1}^{\infty} \exp(-n^2/t) = o(t^R)$, for any $R > 0$, we can rewrite (1.6) as

$$\sum_{i=0}^{\infty} a_i t^{i/2} = 1 + \int_0^t \frac{1 + o(t^R)}{\sqrt{\pi}\sqrt{t-s}} F\left(\sum_{i=0}^{\infty} a_i s^{i/2}, \sum_{i=0}^{\infty} b_i s^{i/2}\right) ds,$$

$$\sum_{i=0}^{\infty} a_i t^{i/2} = 1 + \int_0^t F\left(\sum_{i=0}^{\infty} a_i s^{i/2}, \sum_{i=0}^{\infty} b_i s^{i/2}\right) ds.$$

Starting from the lowest order of t (i.e., $O(t^0)$), by balancing the same order terms on both sides of the above equations we can solve a_i and b_i $(i = 0, 1, 2, \dots)$ iteratively to obtain

$$\phi_1(t) = 1 - \frac{2Cm}{\sqrt{\pi}} t^{1/2} + C^2 m^2 t + \frac{4C^2 m}{3\sqrt{\pi}} (1 + L - Cm^2) t^{3/2} + O(t^2), \tag{2.1a}$$

$$\phi_2(t) = 1 - Cmt + \frac{4C^2 m^2}{3\sqrt{\pi}} t^{3/2} + O(t^2). \tag{2.1b}$$

Furthermore, using (1.8) we have, for small t,

$$F(\phi_1(t), \phi_2(t)) = -Cm + \frac{2C^2m^2}{\sqrt{\pi}} t^{1/2} + C^2m(1 + L - Cm^2)t + \frac{4C^3m^2}{3\sqrt{\pi}} (Cm^2 - 2L - \tfrac{7}{2}) t^{3/2} + \mathrm{O}(t^2), \tag{2.2}$$

which means that any direct application of a product integration method to (1.6) will yield only order $\frac{1}{2}$ global convergence (see e.g. [3,18]). Methods of coping with nonsmoothness of solutions have been studied by many authors; for example Brunner [2] suggested nonpolynomial spline collocation methods, and Norbury and Stuart [15, 16] (and, more recently, Diogo et al. [7]) studied the idea of applying a transformation to the variable of the integration. Here we consider another approach, i.e., subtracting out the singular parts of the integrand. This relatively straightforward method appears to have been first put forward by Eggermont [9] to solve a numerical example.

2.2. Numerical schemes

For clarity of exposition, we present the explicit product Euler scheme, implicit product Euler scheme and product trapezoidal scheme for the system of Volterra integral equations (1.6); higher order schemes can be obtained in a similar fashion. In the derivation of the following numerical schemes the essential idea is to rewrite the integral equation system (1.6) as a different, but equivalent integral equation system in which the singularities have been removed from the integrand.

For $t > 0$, define

$$\kappa(t) = \frac{1}{\sqrt{\pi}} \left(1 + 2 \sum_{n=1}^{\infty} \exp(-n^2/t) \right).$$

This is clearly a bounded function.

To derive order one and order two product integration schemes, consider the following system for $q = 1, 2$:

$$\phi_1(t) = f_1^{(q)}(t) + \int_0^t \frac{G_1^{(q)}(t, s, \phi_1(s), \phi_2(s))}{\sqrt{t-s}}\, \mathrm{d}s, \tag{2.3a}$$

$$\phi_2(t) = f_2^{(q)}(t) + \int_0^t G_2^{(q)}(t, s, \phi_1(s), \phi_2(s))\, \mathrm{d}s, \tag{2.3b}$$

with

$$G_1^{(q)}(t, s, \phi_1(s), \phi_2(s)) = \kappa(t-s)(F(\phi_1(s), \phi_2(s)) + g^{(q)}(s)), \tag{2.4a}$$

$$G_2^{(q)}(t, s, \phi_1(s), \phi_2(s)) = F(\phi_1(s), \phi_2(s)) + g^{(q)}(s), \tag{2.4b}$$

$$f_1^{(q)}(t) = 1 - \int_0^t \kappa(t-s) \frac{g^{(q)}(s)}{\sqrt{t-s}}\, \mathrm{d}s, \tag{2.4c}$$

$$f_2^{(q)}(t) = 1 - \int_0^t g^{(q)}(s)\, \mathrm{d}s, \tag{2.4d}$$

where $g^{(q)}(t)$ are defined using the singular terms in the asymptotic expansion (2.2) such that $F(\phi_1(t), \phi_2(t)) + g^{(q)}(t)$ are q times continuously differentiable,

$$g^{(1)}(t) = -\frac{2}{\sqrt{\pi}} C^2 m^2 t^{1/2}, \tag{2.5}$$

and

$$g^{(2)}(t) = -\frac{2}{\sqrt{\pi}} C^2 m^2 t^{1/2} - \frac{4}{3\sqrt{\pi}} C^3 m^2 (Cm^2 - 2L - \tfrac{7}{2}) t^{3/2}. \tag{2.6}$$

Simple calculations show that both (2.3) with (2.4) and (2.5), and (2.3) with (2.4) and (2.6) are equivalent to system (1.6); furthermore, functions $G_1^{(q)}(t, s, \phi_1(s), \phi_2(s))$ and $G_2^{(q)}(t, s, \phi_1(s), \phi_2(s))$ $(q = 1, 2)$ are q times continuously differentiable with respect to s.

Let ϕ_1^i and ϕ_2^i denote the approximate solutions of $\phi_1(t_i)$ and $\phi_2(t_i)$ respectively on the grids

$$t_i = ih, \quad i = 0, 1, \ldots, N, \qquad Nh = T \ (T > 0).$$

Then we have the following system of product integration schemes:

$$\phi_1^i = \tilde{f}_1^{(q)}(t_i) + \sum_{j=0}^{i} \alpha_{ij}^{(q)} \tilde{G}_1^{(q)}(t_i, t_j, \phi_1^j, \phi_2^j), \tag{2.7a}$$

$$\phi_2^i = f_2^{(q)}(t_i) + \sum_{j=0}^{i} \beta_{ij}^{(q)} G_2^{(q)}(t_i, t_j, \phi_1^j, \phi_2^j), \tag{2.7b}$$

$$\phi_1^0 = 1, \qquad \phi_2^0 = 1, \tag{2.7c}$$

$$i = 1, 2, \ldots, N, \quad q = 1, 2,$$

where

$$\tilde{G}_1^{(q)}(t, s, u, v) = \kappa_l(t - s)(F(u, v) + g^{(q)}(s)) \tag{2.8}$$

with

$$\kappa_l(t) = \frac{1}{\sqrt{\pi}} \left(1 + 2 \sum_{n=1}^{l} \exp(-n^2/t)\right)$$

(l is chosen according to the accuracy required, see Section 3); and $\tilde{f}_1^{(q)}(t_i)$ is the qth order product integration approximation of $f_1^{(q)}(t_i)$, with

$$\int_{t_j}^{t_{j+1}} \frac{s^{p-1/2}}{\sqrt{t_i - s}} \, ds \quad (p = 1, \ldots, q)$$

being calculated analytically, and $\kappa(t_{i-j})$ being approximated by $\kappa_l(t_{i-j})$.

When

$$\alpha_{ij}^{(1)} = \int_{t_j}^{t_{j+1}} \frac{ds}{\sqrt{t_i - s}}, \quad 0 \leqslant j \leqslant i - 1 \leqslant N - 1, \tag{2.9a}$$

$$\alpha_{ii}^{(1)} = \beta_{ii}^{(1)} = 0, \quad 1 \leqslant i \leqslant N, \tag{2.9b}$$

$$\beta_{ij}^{(1)} = h, \quad 0 \leqslant j \leqslant i \leqslant N, \tag{2.9c}$$

we obtain the explicit product Euler scheme, while when

$$\alpha_{ij}^{(1)} = \int_{t_{j-1}}^{t_j} \frac{ds}{\sqrt{t_i - s}}, \quad 1 \leqslant j \leqslant i \leqslant N-1, \tag{2.10a}$$

$$\alpha_{i0}^{(1)} = \beta_{i0}^{(1)} = 0, \quad 1 \leqslant i \leqslant N, \tag{2.10b}$$

$$\beta_{ij}^{(1)} = h, \quad 0 \leqslant j \leqslant i \leqslant N, \tag{2.10c}$$

we obtain the implicit product Euler scheme, and when

$$\alpha_{i0}^{(2)} = \frac{1}{h} \int_0^h \frac{h-s}{\sqrt{t_i - s}}\, ds, \quad 1 \leqslant i \leqslant N, \tag{2.11a}$$

$$\alpha_{ij}^{(2)} = \frac{1}{h} \int_{t_{j-1}}^{t_j} \frac{s - t_{j-1}}{\sqrt{t_i - s}}\, ds + \frac{1}{h} \int_{t_j}^{t_{j+1}} \frac{t_{j+1} - s}{\sqrt{t_i - s}}\, ds, \quad 1 \leqslant j < i \leqslant N, \tag{2.11b}$$

$$\alpha_{ii}^{(2)} = \frac{1}{h} \int_{t_{i-1}}^{t_i} \frac{s - t_{i-1}}{\sqrt{t_i - s}}, \quad 1 \leqslant i \leqslant N, \tag{2.11c}$$

$$\beta_{i0}^{(2)} = \beta_{ii}^{(2)} = \tfrac{1}{2}h, \quad 1 \leqslant i \leqslant N, \tag{2.11d}$$

$$\beta_{ij}^{(2)} = h, \quad 1 \leqslant j < i < N, \tag{2.11e}$$

we obtain the product trapezoidal scheme. For implicit schemes, a system of two nonlinear equations is required to be solved iteratively using Newton's method at every time step. We shall prove in Section 3 that when $h \to 0$, the numerical scheme (2.7) uniquely defines a sequence (ϕ_1^i, ϕ_2^i) $(i = 1, 2, \ldots, N)$ which converges to the exact solution $(\phi_1(t_i), \phi_2(t_i))$ $(i = 1, 2, \ldots, N)$ and that the implicit and explicit product Euler schemes have order one convergence while the product trapezoidal method has order two.

3. Consistency and convergence

Throughout this paper, C with a subscript will be used to denote a positive constant independent of h.

To carry out the consistency and convergence analysis, we present a lemma and a definition.

Lemma 3.1 (see Dixon [8]). *Let*

$$k_l(t) = \frac{\kappa_l(t)}{\sqrt{t}}, \quad t > 0. \tag{3.1}$$

Then,

$$|k(t) - k_l(t)| \leqslant \left(1 - Z\left(l\sqrt{\frac{2}{t}}\right)\right),$$

where

$$Z(x) = \sqrt{\frac{2}{\pi}} \int_0^x \exp(-\tfrac{1}{2}y^2)\,\mathrm{d}y$$

is the normal distribution function.

From now on, l in the truncation of $k(t)$ will be chosen according to the following rule: fix $C_0 > 0$, choose $l = l(T, h)$ such that

$$|\kappa(t) - \kappa_l(t)| < C_0 h^q, \quad t \in [0, T], \tag{3.2}$$

where when $q = 1$ we refer to the implicit or explicit Euler's scheme, and when $q = 2$ we refer to the trapezoidal product integration scheme. From Lemma 3.1, this is achievable by simply letting

$$\sqrt{T}\left(1 - Z\left(l\sqrt{\frac{2}{T}}\right)\right) < C_0 h^q.$$

Definition. For $\forall d > 0$ and $x \in \mathbb{R}^2$, define $S(x, d) = \{ y \in \mathbb{R}^2 \mid \|y - x\|_\infty \leqslant d\}$ and $S_d = S(0, d)$.

We now state the main results of this paper.

Theorem 3.2. *$\forall \rho > 0$, $\exists h_\rho$, such that when $h < h_\rho$, the numerical scheme (2.7) has a sequence of solution $(\phi_1^i, \phi_2^i) \in S_{M+\rho}$, i.e.,*

$$|\phi_1^i| \leqslant M + \rho, \quad |\phi_2^i| \leqslant M + \rho, \tag{3.3}$$

for $i = 0, 1, \ldots, N$, where

$$M = \max\left(\sup_{0 \leqslant t \leqslant T} |\phi_1(t)|, \sup_{0 \leqslant t \leqslant T} |\phi_2(t)| \right). \tag{3.4}$$

Further the numerical scheme has the following convergence property:

$$e^i \leqslant C_* h^q, \quad i = 0, 1, \ldots, N, \tag{3.5}$$

where

$$e^i = e_1^i + e_2^i, \quad i = 0, 1, \ldots, N,$$

and

$$e_1^i = |\phi_1^i - \phi_1(t_i)|, \quad e_2^i = |\phi_2^i - \phi_2(t_i)|, \qquad i = 0, 1, \ldots, N.$$

The existence of M has been proven in [11]. Separate proofs of Theorem 3.2 will be given for the explicit scheme and the implicit schemes in Section 3.2.

Obviously the explicit Euler scheme (2.7) yields a unique solution. The following theorem shows that the nonlinear system (2.7) is solvable for implicit schemes, i.e., (ϕ_1^i, ϕ_2^i), referred to in Theorem 3.2, can be obtained by using Newton's iteration applied to the nonlinear system (2.7). This theorem will also be proven in Section 3.2.

Theorem 3.3. *Suppose that for the implicit schemes* (2.7) *and for some* $r > 0$, *the starting values of Newton's iteration for the solution of the nonlinear system* (2.7) *at every time step are in* S_r, *then for* $\forall \rho > 0$, $\exists h_* = h_*(T, r, \rho) > 0$, *such that when* $h < h_*$, *the Newton iterates converge to the approximate numerical solution* (ϕ_1^i, ϕ_2^i) $(i = 1, 2, \ldots, N)$, *which is unique in* $S_{M+\rho}$.

3.1. Consistency

Since the numerical schemes (2.7) are all based on the idea of product integration, their consistency arguments are essentially the same.

Denote for $q = 1, 2$ and $i = 1, 2, \ldots, N$, the truncation errors of the scheme (2.7) by δ_1^i and δ_2^i, and the truncation error of the product integration approximation of $f_1^{(q)}(t_i)$ by δ_*^q $(|\delta_*^q| \leqslant C_f h^q)$, then

$$\delta_1^i = \sum_{j=0}^{i} \alpha_{ij}^{(q)} \tilde{G}_1^{(q)}(t_i, t_j, \phi_1(t_j), \phi_2(t_j)) - \int_0^{t_i} \frac{G_1^{(q)}(t_i, s, \phi_1(s), \phi_2(s))}{\sqrt{t-s}} \, \mathrm{d}s + |\delta_*^q|, \tag{3.6a}$$

$$\delta_2^i = \sum_{j=0}^{i} \beta_{ij}^{(q)} G_2^{(q)}(t_i, t_j, \phi_1(t_j), \phi_2(t_j)) - \int_0^{t_i} G_2^{(q)}(t_i, s, \phi_1(s), \phi_2(s)) \, \mathrm{d}s. \tag{3.6b}$$

Obviously

$$\begin{aligned} |\delta_1^i| \leqslant & \left| \sum_{j=0}^{i} \alpha_{ij}^{(q)} (\tilde{G}_1^{(q)}(t_i, t_j, \phi_1(t_j), \phi_2(t_j)) - G_1^{(q)}(t_i, t_j, \phi_1(t_j), \phi_2(t_j))) \right| \\ & + \left| \sum_{j=0}^{i} \alpha_{ij}^{(q)} G_1^{(q)}(t_i, t_j, \phi_1(t_j), \phi_2(t_j)) - \int_0^{t_i} \frac{G_1^{(q)}(t_i, s, \phi_1(s), \phi_2(s))}{\sqrt{t-s}} \, \mathrm{d}s \right| + |\delta_*^q| \\ := & \; I_1^i + I_2^i + |\delta_*^q|. \end{aligned}$$

Noting that for $q = 1, 2$,

$$\sum_{j=0}^{i} \alpha_{ij}^{(q)} = \int_0^{t_i} \frac{\mathrm{d}s}{\sqrt{t_i - s}} = 2\sqrt{t_i} \leqslant 2\sqrt{T}, \qquad \sum_{j=0}^{i} \beta_{ij}^{(q)} = t_i \leqslant T, \tag{3.7a,b}$$

we have from (2.8) and (3.2)

$$I_1^i \leqslant \sum_{j=0}^{i} \alpha_{ij}^{(q)} | F(\phi_1(t_j), \phi_2(t_j)) + g^{(q)}(t_j) || \kappa(t_i - t_j) - \kappa_l(t_i - t_j)| \leqslant 2C_0 \sqrt{T} M^{(q)} h^q, \tag{3.8}$$

where

$$M^{(q)} = \sup_{(x,y) \in S_M} | F(x, y)| + \sup_{t \in [0,T]} | g^{(q)}(t)|. \tag{3.9}$$

Since $G_1^{(q)}(t,s,\phi_1(s),\phi_2(s)) \in C^q[0,T]$, we obtain (see e.g. [5])

$$I_2^i \leqslant C_1 h^q. \tag{3.10}$$

Combining (3.8) and(3.10) shows that $|\delta_1^i| \leqslant C_2 h^q$. Since $G_2^{(q)}(t,s,\phi_1(s),\phi_2(s)) \in C^q[0,T]$, we have $|\delta_2^i| \leqslant C_3 h^q$ (see [5]).

3.2. *Convergence*

We cannot directly employ the standard method of proving convergence of product integration schemes for Volterra–Abel equations of the second kind (see e.g. [5]), because the nonlinearity of the function F involved in (1.6) does not allow a global Lipschitz constant. However F is locally Lipschitz continuous, because F is continuously differentiable with respect to both of its variables.

3.2.1. *The explicit Euler scheme*

Proof of Theorem 3.2. Firstly, we shall inductively prove that, for a given $\rho > 0$, when h is sufficiently small, (ϕ_1^i, ϕ_2^i) $(i = 0,1, \dots, N)$ obtained by using the numerical scheme (2.7) with $\alpha_{ij}^{(1)}$, $\beta_{ij}^{(1)}$, defined by (2.9), satisfy (3.3).

For $i = 0$, (3.3) and (3.5) are obvious.

Suppose when $h < h_1$,

$$|\phi_1^i| \leqslant M + \rho, \quad |\phi_2^i| \leqslant M + \rho, \qquad i = 0,1, \dots, k-1 \quad (k > 2).$$

Then by subtracting (2.7a) from (2.3a), and (2.7b) from (2.3b) respectively with $t = t_i$, and using (3.6) we have for $i = 1,2, \dots, k$,

$$e_1^i \leqslant \left| \sum_{j=0}^{i-1} \alpha_{ij}^{(1)} (\tilde{G}_1^{(1)}(t_i, t_j, \phi_1(t_j), \phi_2(t_j)) - \tilde{G}_1^{(1)}(t_i, t_j, \phi_1^j, \phi_2^j)) \right| + |\delta_1^i|, \tag{3.11a}$$

$$e_2^i \leqslant \left| \sum_{j=0}^{i-1} \beta_{ij}^{(1)} (G_2^{(1)}(t_i, t_j, \phi_1(t_j), \phi_2(t_j)) - G_2^{(1)}(t_i, t_j, \phi_1^j, \phi_2^j)) \right| + |\delta_2^i|. \tag{3.11b}$$

From (2.9), (2.10) and (2.11), for $h > 0$ and $q = 1,2$,

$$\alpha_{ij}^{(q)} \leqslant \frac{C_4 h^{1/2}}{\sqrt{i-j}}, \; \alpha_{ii}^{(q)} \leqslant C_4 h^{1/2}, \quad 0 \leqslant j < i \leqslant N, \tag{3.12a}$$

$$\beta_{ij}^{(q)} \leqslant \frac{C_4 h^{1/2}}{\sqrt{i-j}}, \; \beta_{ii}^{(q)} \leqslant C_4 h^{1/2}, \quad 0 \leqslant j < i \leqslant N. \tag{3.12b}$$

Define L_ρ as a local Lipschitz constant of F with respect to both of its variables on $S_{M+\rho}$, i.e., $\forall x = (x_1, x_2) \in S_{M+\rho}$, and $\forall y = (y_1, y_2) \in S_{M+\rho}$,

$$|F(x_1, x_2) - F(y_1, y_2)| \leqslant L_\rho \sum_{n=1}^{2} |x_n - y_n|. \tag{3.13}$$

Thus by using (2.8) and (3.12a), we have from (3.11a), for $h < h_1$ and $i = 1, 2, \ldots, k$,

$$e_1^i \leqslant \sum_{j=0}^{i-1} \alpha_{ij}^{(1)} |F(\phi_1(t_j), \phi_2(t_j)) - F(\phi_1^j, \phi_2^j)| \kappa_l(t_i - t_j) + |\delta_1^i|$$

$$\leqslant C_4 L_\rho M_\kappa h^{1/2} \sum_{j=0}^{i-1} \frac{e^j}{\sqrt{i-j}} + |\delta_1^i|, \tag{3.14a}$$

where

$$M_\kappa = \sup_{0 \leqslant t \leqslant T} |\kappa(t)| \geqslant \sup_{0 \leqslant t \leqslant T} |\kappa_l(t)|.$$

Likewise using (2.4b) and (3.12b), we can prove from (3.11b) that, for $h > 0$,

$$e_2^i \leqslant C_4 L_\rho h^{1/2} \sum_{j=0}^{i-1} \frac{e^j}{\sqrt{i-j}} + |\delta_2^i|, \quad i = 1, 2, \ldots, k. \tag{3.14b}$$

Adding (3.14a) and (3.14b) together yields

$$e^i \leqslant C_4 L_\rho (M_\kappa + 1) h^{1/2} \sum_{j=0}^{i-1} \frac{e^j}{\sqrt{i-j}} + |\delta^i|, \quad i = 1, 2, \ldots, k, \tag{3.15}$$

where from the consistency proved previously,

$$|\delta^i| = |\delta_1^i| + |\delta_2^i| \leqslant C_5 h, \quad i = 1, 2, \ldots, N. \tag{3.16}$$

Hence by employing the discrete Gronwall inequality (see e.g. [5]) to (3.15) we obtain a $C_6 > 0$ independent of k, such that when $h < h_1$,

$$e^i \leqslant C_6 h, \quad i = 1, 2, \ldots, k. \tag{3.17}$$

Concluding the induction we obtain, for $h < h_2 = \min(\rho/C_6, h_1)$,

$$|\phi_1^i| \leqslant M + \rho, \quad |\phi_2^i| \leqslant M + \rho, \qquad i = 0, 1, \ldots, N.$$

This allows us to repeat the arguments (3.11)–(3.17) for $i = 1, 2, \ldots, N$ to derive, when $h < h_2$,

$$e^i \leqslant C_6 h, \quad i = 0, 1, \ldots, N. \quad \square$$

3.2.2. The implicit schemes

For clarity of exposition, we prove Theorems 3.2 and 3.3 for the trapezoidal scheme. The argument for the implicit Euler scheme is similar.

To prove Theorem 3.2 we need the following lemma.

Lemma 3.4. (see e.g. Ortega and Rheinbolt [17]). *Suppose that, for the nonsingular matrix A, the continuous function $U: D \subset \mathbb{R}^2 \to \mathbb{R}^2$ satisfies*

$$\|U(x) - U(y)\| \leqslant \alpha \|x - y\|, \quad \forall\, x, y \in D_1,$$

where $D_1 = S(x_0, \delta) \subset D$, $0 < \alpha < \beta^{-1}$, and $\beta = \|A^{-1}\|$, then $V(x) = Ax - U(x)$ is a homeomorphism between D_1 and $V(D_1)$. Moreover, for any $y \in D_2 = S(V(x_0), \sigma)$, where $\sigma = (\beta^{-1} - \alpha)\delta$, the equation $V(x) = y$ has a unique solution in D_1. Hence, in particular there is $D_2 \subset V(D_1)$.

Proof of Theorem 3.2. ($q = 2$) Let $\phi^i = (\phi_1^i, \phi_2^i)$, then using (2.8), (2.4b), and the identity

$$\kappa_l(0) = \frac{1}{\sqrt{\pi}},$$

the product trapezoidal scheme (2.7) can be written as

$$P_i(\phi^i) = \begin{pmatrix} X_i(\phi^i) \\ Y_i(\phi^i) \end{pmatrix} = \begin{pmatrix} A_i \\ B_i \end{pmatrix}, \quad i = 1, 2, \ldots, N, \tag{3.18}$$

where

$$X_i(\phi^i) = \phi_1^i - \frac{\alpha_{ii}^{(2)}}{\sqrt{\pi}} F(\phi_1^i, \phi_2^i), \tag{3.19a}$$

$$Y_i(\phi^i) = \phi_2^i - \beta_{ii}^{(2)} F(\phi_1^i, \phi_2^i), \tag{3.19b}$$

and

$$A_i = \sum_{j=0}^{i-1} \alpha_{ij}^{(2)} \tilde{G}_1^{(2)}(t_i, t_j, \phi_1^j, \phi_2^j) + \tilde{f}_1^{(2)}(t_i) + \frac{\alpha_{ii}^{(2)}}{\sqrt{\pi}} g^{(2)}(t_i), \tag{3.20a}$$

$$B_i = \sum_{j=0}^{i-1} \beta_{ij}^{(2)} G_2^{(2)}(t_i, t_j, \phi_1^j, \phi_2^j) + f_2^{(2)}(t_i) + \beta_{ii}^{(2)} g^{(2)}(t_i). \tag{3.20b}$$

Since, from (3.18), (3.19) and (1.8), for $x = (\xi, \eta) \in \mathbb{R}^2$, the Fréchet differentiation of P_i can be written as

$$P_i'(x) = I + Q_i(x), \tag{3.21}$$

where I is the unit matrix, and

$$Q_i(x) = C \begin{pmatrix} \dfrac{\alpha_{ii}^{(2)}}{\sqrt{\pi}} \eta & \dfrac{\alpha_{ii}^{(2)}}{\sqrt{\pi}} \xi \\ \beta_{ii}^{(2)} \eta & \beta_{ii}^{(2)} \xi \end{pmatrix} + C \begin{pmatrix} \dfrac{\alpha_{ii}^{(2)}}{\sqrt{\pi}} (m-1) & \dfrac{\alpha_{ii}^{(2)}}{\sqrt{\pi}} L \\ \beta_{ii}^{(2)} (m-1) & \beta_{ii}^{(2)} L \end{pmatrix}. \tag{3.22}$$

From (3.12),

$$\|Q_i(x)\|_\infty \leqslant C_7 h^{1/2}, \quad x \in S_{M+\rho}, \tag{3.23}$$

so by applying Banach's inequality to (3.21) we see that, for $R_1 > 0, \exists h_3 > 0$, such that when $h < h_3$,

$$\|(P_i'(x))^{-1}\|_\infty \leqslant \frac{1}{1 - \|Q_i(x)\|_\infty} \leqslant R_1 \quad \forall x \in S_{M+\rho}. \tag{3.24}$$

Again using (3.12) we have, $\forall x = (\xi, \eta) \in \mathbb{R}^2$,

$$\|P_i''(x)\|_\infty = \max\left(\left|\frac{\partial^2 X_i}{\partial \xi^2}\right| + \left|\frac{\partial^2 X_i}{\partial \xi \partial \eta}\right| + \left|\frac{\partial^2 X_i}{\partial \eta^2}\right|, \left|\frac{\partial^2 Y_i}{\partial \xi^2}\right| + \left|\frac{\partial^2 Y_i}{\partial \xi \partial \eta}\right| + \left|\frac{\partial^2 Y_i}{\partial \eta^2}\right|\right)$$

$$= \max\left\{\left|\frac{\partial^2 X_i}{\partial \xi \partial \eta}\right|, \left|\frac{\partial^2 Y_i}{\partial \xi \partial \eta}\right|\right\} = \max\left(\frac{C\alpha_{ii}^{(2)}}{\sqrt{\pi}}, C\beta_{ii}^{(2)}\right) \leqslant C_8 h^{1/2}, \tag{3.25}$$

so for any $x_0 \in S_M$, letting $W_i = P_i'(x_0)$ and $H_i(x) = W_i x - P_i(x)$ shows that for $i = 1, 2, \ldots, N$, $\forall x \in S_{M+\rho}$ and $h > 0$,

$$\begin{aligned}\|H_i(x) - H_i(x_0)\|_\infty &= \|P_i(x) - P_i(x_0) - W_i(x - x_0)\|_\infty \\ &\leqslant \sup_{0 \leqslant t \leqslant 1} \|P'(x_0 + t(x - x_0)) - P'(x_0)\|_\infty \|x - x_0\|_\infty \\ &\leqslant \sup\{\|P_i''(x')\|_\infty \|x - x_0\|_\infty^2 \,|\, x' \in S_{M+\rho}\} \\ &\leqslant 2C_8(M + \rho)h^{1/2}\|x - x_0\|_\infty;\end{aligned} \tag{3.26}$$

here the mean value theorem was employed twice.

Take $R_2 \in (0, 1/R_1)$, and $h_4 \in (0, h_3)$ such that $2C_8(\rho + M)h_4^{1/2} < R_2$. So from Lemma 3.4, for $i = 1, 2, \ldots, N$ and $h < h_4$, $W_i - H_i = P_i : S(x_0, \rho) \to P_i(S(x_0, \rho))$ is a homeomorphism, and

$$(P_i)^{-1}(S(P_i(x_0), \sigma_\rho)) \subset S(x_0, \rho) \subset S_{M+\rho}, \quad i = 1, 2, \ldots, N, \tag{3.27}$$

where

$$\sigma_\rho = (1/R_1 - R_2)\rho. \tag{3.28}$$

Since (3.27) is true $\forall x_0$ in S_M, and δ_ρ is independent of x_0, summarizing the analysis (3.18)–(3.28) we can state: $\forall \rho > 0$, $\exists \delta_\rho > 0$ and $h_4 = h_4(T, \rho) > 0$, such that for $h < h_4$ and $i = 1, 2, \ldots, N$,

$$(P_i)^{-1}(S(P_i(x), \sigma_\rho)) \subset S_{M+\rho}, \quad \forall x \in S_M. \tag{3.29}$$

Now we shall use the result (3.29) in the following inductive argument to prove that for the trapezoidal scheme, (3.3) holds for $i = 0, 1, \ldots, N$.

Given $\rho > 0$, (3.3) and (3.5) are obvious for $i = 0$.

Suppose that when $h < h_5$ (3.3) is true for $i = 0, 1, \ldots, k - 1$ ($k > 2$). Following the arguments used for the explicit scheme, we have for $i = 1, 2, \ldots, k - 1$,

$$e_1^i \leqslant \left|\sum_{j=0}^{i} \alpha_{ij}^{(2)}(\tilde{G}_1^{(2)}(t_i, t_j, \phi_1(t_j), \phi_2(t_j)) - \tilde{G}_1^{(2)}(t_i, t_j, \phi_1^j, \phi_2^j))\right| + |\delta_1^i|, \tag{3.30a}$$

$$e_2^i \leqslant \left|\sum_{j=0}^{i} \beta_{ij}^{(2)}(G_2^{(2)}(t_i, t_j, \phi_1(t_j), \phi_2(t_j)) - G_2^{(2)}(t_i, t_j, \phi_1^j, \phi_2^j))\right| + |\delta_2^i|, \tag{3.30b}$$

Since $\phi^j = (\phi_1^j, \phi_2^j) \in S_{M+\rho}$ $(j = 0, 1, \dots, k-1)$, we can employ arguments similar to those used for (3.11)–(3.15) to obtain

$$e^i \leqslant C_4 L_\rho (M_\kappa + 1) h^{1/2} \sum_{j=0}^{i-1} \frac{e^j}{\sqrt{i-j}} + C_4 L_\rho (M_\kappa + 1) h^{1/2} e^i + |\delta^i|, \quad i = 1, 2, \dots, k-1. \tag{3.31}$$

Hence there exists an $h_6 \in (0, h_5)$, such that when $h < h_6$,

$$e^i \leqslant \frac{C_4 L_\rho (M_\kappa + 1) h^{1/2}}{1 - C_4 L_\rho (M_\kappa + 1) h^{1/2}} \sum_{j=0}^{i-1} \frac{e^j}{\sqrt{i-j}} + \frac{|\delta^i|}{1 - C_4 L_\rho (M_\kappa + 1) h^{1/2}}, \quad i = 1, 2, \dots, k-1. \tag{3.32}$$

From (3.16), $\delta^i < C_5 h^2$, so by applying the discrete Gronwall inequality (see e.g. [5]) to (3.32) we obtain

$$e^i \leqslant C_9 h^2, \quad i = 0, 1, \dots, k-1. \tag{3.33}$$

Obviously from (2.3) and (3.6), $(\phi_1(t_k), \phi_2(t_k))$ is the solution of

$$P_i(x) = \begin{pmatrix} \tilde{A}_k \\ \tilde{B}_k \end{pmatrix}, \tag{3.34}$$

where

$$\tilde{A}_k = \sum_{j=0}^{k-1} \alpha_{kj}^{(2)} \tilde{G}_1^{(2)}(t_k, t_j, \phi_1(t_j), \phi_2(t_j)) + \tilde{f}_1^{(2)}(t_k) + \frac{\alpha_{kk}^{(2)}}{\sqrt{\pi}} g(t_k) - \delta_1^k, \tag{3.35a}$$

$$\tilde{B}_k = \sum_{j=0}^{k-1} \beta_{kj}^{(2)} G_2^{(2)}(t_k, t_j, \phi_1(t_j), \phi_2(t_j)) + f_2^{(2)}(t_k) + \beta_{kk}^{(2)} g(t_k) - \delta_2^k, \tag{3.35b}$$

and δ_1^k and δ_2^k are the truncation errors satisfying $\delta_1^k < C_2 h^2$ and $\delta_2^k < C_3 h^2$. Subtracting (3.20a) from (3.35a), and using (2.8) and (3.13) yields

$$\begin{aligned} |\tilde{A}_k - A_k| &\leqslant \left| \sum_{j=0}^{k-1} \alpha_{kj}^{(2)} (\tilde{G}_1^{(2)}(t_k, t_j, \phi_1(t_j), \phi_2(t_j)) - \tilde{G}_1^{(2)}(t_i, t_j, \phi_1^j, \phi_2^j)) \right| + |\delta_1^k| \\ &\leqslant \sum_{j=0}^{k-1} \alpha_{kj}^{(2)} |F(\phi_1(t_j), \phi_2(t_j)) - F(\phi_1^j, \phi_2^j)| \kappa_l(t_k - t_j) + |\delta_1^i| \\ &\leqslant L_\rho M_\kappa \sum_{j=0}^{k-1} \alpha_{kj}^{(2)} e^j + C_2 h^2, \end{aligned}$$

to which employing (3.7) and (3.33) admits $|\tilde{A}_k - A_k| \leqslant C_{10}h^2$. Likewise, $|\tilde{B}_k - B_k| \leqslant C_{11}h^2$. Hence we obtain $h_7 \in (0, h_6)$, such that when $h < h_7$,

$$\max\{|\tilde{A}_k - A_k|, \quad |\tilde{B}_k - B_k|\} \leqslant \delta_\rho, \tag{3.36}$$

where δ_ρ is defined in (3.28). Furthermore, comparing (3.18) with (3.34) and using (3.29) show that when $h < h_8 = \min(h_7, h_4)$, (2.7) with $q = 2$ and $i = k$ has a solution $\phi^k = (\phi_1^k, \phi_2^k) \in S_{M+\rho}$.

To conclude the induction: $\forall \rho > 0$, $\exists h_8 = h_8(T, \rho) > 0$, such that when $h < h_8$, the numerical scheme (2.7) with $q = 2$ has a sequence of solution $\phi^i = (\phi_1^i, \phi_2^i) \in S_{M+\rho}$, for $i = 1, 2, \ldots, N$.

The result of the induction enables us to repeat the arguments (3.30)–(3.33) for $h < h_8$ and $i = 1, 2, \ldots, N$ to show: when $h < h_8$, $e^i \leqslant C_9 h^2$, $i = 1, 2, \ldots, N$. The proof of Theorem 3.2 is thus complete. □

To prove Theorem 3.3, we require the following lemma.

Lemma 3.5 (Newton–Kantorovich Theorem, see e.g. Ortega and Rheinbolt [17]). *Suppose the function U is Fréchet differentiable on* $\mathbb{R}^2$ *and*

$$\|U'(x) - U'(y)\| \leqslant \mu\|x - y\|, \quad x, y \in \mathbb{R}^2.$$

Also for $x_0 \in \mathbb{R}^2$, $\|(U'(x_0))^{-1}\| \leqslant \alpha$, $\|(U'(x_0))^{-1}U(x_0)\| \leqslant \beta$, *and* $\gamma = \mu\alpha\beta \leqslant \frac{1}{2}$. *Then the Newton iterates* $x_{i+1} = x_i - (U'(x_i))^{-1}U(x_i)$, $i = 0, 1, \ldots,$ *are well defined on* $S(x_0, d)$, *and converge to a solution of* $U(x) = 0$ *which is unique in* $S(x_0, a)$, *where* $a = [(1 + \sqrt{1 - 2\gamma})/\gamma]\beta$.

Proof of Theorem 3.3. ($q = 2$) Theorem 3.2 states that, $\forall \rho > 0$, $\exists h_\rho > 0$ such that when $h < h_\rho$, (2.7) has a sequence of solutions $\phi^i = (\phi_1^i, \phi_2^i) \in S_{M+\rho}$ $(i = 1, 2, \ldots, N)$. Since $\tilde{G}_1^{(2)}$, $\tilde{G}_2^{(2)}$, $f_1^{(2)}$ and $f_2^{(2)}$ are continuous, in (3.20),

$$|A_i| \leqslant C_{12}, \quad |B_i| \leqslant C_{12}, \qquad i = 1, 2, \ldots, N. \tag{3.37}$$

Rewriting the system of nonlinear equations (3.18) as

$$J_i(\phi^i) = \begin{pmatrix} X_i(\phi^i) - A_i \\ Y_i(\phi^i) - B_i \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad i = 1, 2, \ldots, N, \tag{3.38}$$

yields for $h < h_\rho$ and $x \in S_r$,

$$\|J_i(x)\|_\infty \leqslant C_{13}, \quad i = 1, 2, \ldots, N. \tag{3.39}$$

And from (3.21) and (3.22), $\forall x, y \in \mathbb{R}^2$,

$$\begin{aligned} \|J_i'(x) - J_i'(y)\|_\infty &= \|Q_i(x) - Q_i(y)\|_\infty \\ &= \|Q_i(x - y)\|_\infty \leqslant C_{14}h^{1/2}\|x - y\|_\infty. \end{aligned} \tag{3.40}$$

Repeating arguments similar to those used for (3.21)–(3.24) shows that there exist an $h_9 \in (0, h_\rho)$, such that for $h < h_9$ and $\forall x \in S_r$,

$$\|(J_i'(x))^{-1}\|_\infty \leqslant C_{15}, \quad i = 1, 2, \ldots, N. \tag{3.41}$$

Thus using (3.39) we have, $\forall x \in S_r$,

$$\|(J_i'(x))^{-1} J_i(x)\|_\infty \leqslant \|(J_i'(x))^{-1}\|_\infty \|J_i(x)\|_\infty \leqslant C_{13}C_{15}, \quad i = 1, 2, \dots, N.$$

Suppose $y_i \in S_r$ $(i = 1, 2, \dots, N)$ is the starting value of the Newton iteration at the ith time step, then from Lemma 3.5 there exists an $h_{10} \in (0, h_9)$ such that when $h < h_{10}$, the iterates converge to the unique solution of (3.18) in $S(y_i, r^*)$, where

$$r^* = \frac{1 + \sqrt{1 - 2C_{14}h^{1/2}}}{C_{14}h^{1/2}}\, C_{13}C_{15}.$$

Hence there exists an $h_{11} \in (0, h_{10})$ independent of y_i, such that when $h < h_{11}$,

$$S(y_i, r^*) \supset S_{M+\rho}, \quad i = 1, 2, \dots, N.$$

In conclusion we see that, $\forall r > 0, \exists h_r \in (0, h_\rho)$, such that when $h < h_r$, Newton's iteration applied to the nonlinear system of equations (3.18) with the starting value y_i ($\forall\, y_i \in S_r$), gives the numerical solution $\phi^i = (\phi_1^i, \phi_2^i)$ $(i = 1, 2, \dots, N)$, which is unique in $S_{M+\rho}$. Hence the proof of Theorem 3.3 is also complete. □

4. Numerical examples

Numerical calculations were performed for the case $E = 0.2$, $L = 0.01$, $m = 1.0$ and $l = 8$ (which assures 10^{-11} accuracy for the approximation of $k(t)$ on $(0 < t < 1)$) using three numerical schemes, i.e., the implicit and explicit Euler product integration methods and the product trapezoidal method. Algorithms are implemented using double precision FORTRAN 77. The approximate solutions of $u(1,\ 0.05)$, $u(1,\ 0.5)$, $\gamma(0.05)$ and $\gamma(0.5)$ obtained using the product integration schemes (2.7) are shown in Tables 1–3. The differences of these approximate solutions for consecutive values of the step size h are also presented. These differences (denoted by Δ in the tables) indicate convergence of order 1 for the explicit and implicit Euler product integration schemes, and of order 2 for the trapezoidal product integration method (see [12] for Aitken's method).

Table 1
The explicit Euler scheme

h, Δ	$u(1, 0.05)$	$\gamma(0.05)$	$u(1, 0.5)$	$\gamma(0.5)$
$h = \frac{1}{40}$	0.952130	0.0095527	0.86062	0.085823
Δ	5.5D − 5	− 8.7D − 6	3.9D − 4	− 5.2D − 5
$h = \frac{1}{80}$	0.952185	0.0095440	0.86101	0.085771
Δ	2.6D − 5	− 4.2D − 6	1.9D − 4	− 2.6D − 5
$h = \frac{1}{160}$	0.952211	0.0095398	0.86120	0.085745

Table 2
The implicit Euler scheme

h, Δ	$u(1, 0.05)$	$\gamma(0.05)$	$u(1, 0.5)$	$\gamma(0.5)$
$h = \frac{1}{40}$	0.952294	0.0095192	0.86208	0.085616
Δ	$-2.7D-5$	$8.2D-6$	$-3.4D-4$	$5.2D-5$
$h = \frac{1}{80}$	0.952267	0.0095274	0.86174	0.085668
Δ	$-1.5D-5$	$4.1D-6$	$-1.8D-4$	$2.6D-5$
$h = \frac{1}{160}$	0.952252	0.0095315	0.86156	0.085694

Table 3
The trapezoidal product integration scheme

h, Δ	$u(1, 0.05)$	$\gamma(0.05)$	$u(1, 0.5)$	$\gamma(0.5)$
$h = \frac{1}{40}$	0.952234084	0.0095356151	0.8613730	0.08571871
Δ	$-4.7D-8$	$1.1D-8$	$7.6D-5$	$7.5D-7$
$h = \frac{1}{80}$	0.952234037	0.0095356259	0.8613806	0.08571946
Δ	$-1.2D-8$	$2.8D-9$	$1.9D-6$	$1.9D-7$
$h = \frac{1}{160}$	0.952234025	0.0095356287	0.8613825	0.08571965

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